

QFT over the finite line. Heat kernel coefficients, spectral zeta functions and selfadjoint extensions

J. M. Muñoz-Castañeda^{1*}, Klaus Kirsten^{2†} and M. Bordag^{1‡}

¹*Institut für Theoretische Physik, Universität Leipzig, Leipzig, 04103, Germany.*

²*GCAP-CASPER Department of Mathematics, Baylor University, Waco, TX 76798, USA.*

Abstract

Following the seminal works of Asorey-Ibort-Marmo and Muñoz-Castañeda-Asorey about selfadjoint extensions and quantum fields in bounded domains, we compute all the heat kernel coefficients for any non-negative selfadjoint extension of the Laplace operator over the finite line $[0, L]$. The derivative of the corresponding spectral zeta function at $s = 0$ (partition function of the corresponding quantum field theory) is obtained. In order to compute the correct expression for the $a_{1/2}$ heat kernel coefficient, it is necessary to know in detail which non-negative selfadjoint extensions have zero modes and how many of them they have. This question is also answered in detail.

1 Introduction. Basic formulas and results

The physical system on which we will focus is a free massless scalar quantum field theory defined over the finite interval $[0, L]$. The quantum hamiltonian that describes the one particle states of this quantum field theory is given by the Laplace operator over the finite line $[0, L]$. It is a very well known fact that the Laplace operator over the finite line $[0, L]$ is not an essentially selfadjoint operator but instead admits an infinite set of selfadjoint extensions. Physically speaking this means that there is an infinite set of possible quantum field theories that describe the behaviour of a free quantum massless scalar field confined to propagate in the interval $[0, L]$. In order to respect the unitarity principle of quantum field theory we must only take into account those selfadjoint extensions of the Laplace operator that give rise to non-negative selfadjoint operators (see [1]). As described in [1] among the set of non-negative selfadjoint extensions we can distinguish between two different types:

1. Non-negative selfadjoint extensions of Δ over $[0, L]$ that are non-negative only for certain values of the finite length L of the interval. Typically these selfadjoint extensions are

*jose.munoz-castaneda@uni-leipzig.de

†klaus_kirsten@baylor.edu

‡bordag@itp.uni-leipzig.de

non-negative for $L \geq L_0$ for a given L_0 that depends on the selfadjoint extension. When $L < L_0$ these selfadjoint extensions have negative eigenvalues and thus give rise to non unitary quantum field theories. We will call these selfadjoint extensions *weakly consistent selfadjoint extensions*.

2. Non-negative selfadjoint extensions of Δ over $[0, L]$ that are non-negative for any value of the finite length L of the finite line. These selfadjoint extensions have only zero and positive eigenvalues for any value of $L \in (0, \infty)$. We will call these selfadjoint extensions *strongly consistent selfadjoint extensions* and following [1] we denote by \mathcal{M}_F the set of strongly consistent selfadjoint extension.

In this paper we will focus only on those free massless scalar quantum field theories defined by selfadjoint extensions of Δ over $[0, L]$ that are non-negative for all $L \in (0, \infty)$. Hence we will only be interested in the selfadjoint extensions contained in \mathcal{M}_F . Typically one distinguishes separated and coupled boundary conditions [2], but in the formulation of [1] this will not be necessary.

In order to be able to characterize the selfadjoint extensions of \mathcal{M}_F we will use the Asorey-Ibort-Marmo (AIM) formalism (see [3]) to characterize the selfadjoint extensions¹ of Δ over the finite line $[0, L]$. From the first AIM theorem (see [3, 1]) the set of selfadjoint extensions of Δ over $[0, L]$ is in one-to-one correspondence with the group $U(2)$. Given any $U \in U(2)$ we will denote the corresponding selfadjoint extension by Δ_U . Each selfadjoint extension Δ_U is defined by its domain of functions $\mathcal{D}_U \subset \mathcal{L}^2([0, L], \mathbb{C})$. The domain $\mathcal{D}_U \subset \mathcal{L}^2([0, L], \mathbb{C})$ that defines the selfadjoint extension Δ_U is given in terms of the matrix $U \in U(2)$ (see [3, 1]) by

$$\mathcal{D}_U = \{ \psi \in \mathcal{L}^2([0, L], \mathbb{C}) / \varphi - i\dot{\varphi} = U(\varphi + i\dot{\varphi}) \}, \quad (1.1)$$

where φ and $\dot{\varphi}$ are the boundary data for $\psi \in \mathcal{L}^2([0, L], \mathbb{C})$:

$$\varphi \equiv \begin{pmatrix} \psi(0) \\ \psi(L) \end{pmatrix}, \quad \dot{\varphi} \equiv \begin{pmatrix} -\psi'(0) \\ \psi'(L) \end{pmatrix}. \quad (1.2)$$

Following the notation in [1] for any $\psi \in \mathcal{L}^2([0, L], \mathbb{C})$ we introduce the 2 dimensional column vectors $\varphi_{\pm}(\psi)$:

$$\varphi_{\pm}(\psi) \equiv \begin{pmatrix} \psi(0) \mp i\psi'(0) \\ \psi(L) \pm i\psi'(L) \end{pmatrix}. \quad (1.3)$$

We can write the boundary condition given in eq. (1.1) in terms of $\varphi_{\pm}(\psi)$ as

$$\varphi_{-}(\psi) = U \cdot \varphi_{+}(\psi). \quad (1.4)$$

Following the conventions and notation used in [1] we parameterize the elements $U \in U(2)$ by using 5 parameters:

$$U(\alpha, \beta, \mathbf{n}) = e^{i\alpha} [\cos(\beta)\mathbb{I} + i\sin(\beta)(\mathbf{n} \cdot \boldsymbol{\sigma})], \quad (1.5)$$

where \mathbb{I} is the 2x2 identity matrix, $\boldsymbol{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$ are the Pauli matrices, \mathbf{n} is a 3 dimensional unit vector ($n_1^2 + n_2^2 + n_3^2 = 1$) and the angles α and β are such that

$$\alpha \in [0, \pi]; \quad \beta \in [-\pi/2, \pi/2]. \quad (1.6)$$

¹We will denote by \mathcal{M} the set of all the selfadjoint extensions of Δ over the finite line $[0, L]$.

Using this parametrization we can characterize the non-zero part of the spectrum for any $\Delta_U \in \mathcal{M}$ including multiplicities of eigenvalues (see the consistency lemma in [1]). The secular equation obtained in [1] for any $\Delta_U \in \mathcal{M}$ is given by

$$\begin{aligned} h_U(k) &= 2ie^{i\alpha} [\sin(kL) ((k^2 - 1) \cos(\beta) + (k^2 + 1) \cos(\alpha)) \\ &\quad - 2k \sin(\alpha) \cos(kL) - 2kn_1 \sin(\beta)]. \end{aligned} \quad (1.7)$$

The non-zero part $\tilde{\sigma}(\Delta_U)$ of the spectrum of $\Delta_U \in \mathcal{M}$ is given by

$$\tilde{\sigma}(\Delta_U) = \{k^2 \in \mathbb{R} - \{0\} / h_U(k) = 0\} = \{k^2 \in \mathbb{R} - \{0\} / k \in Z(h_U) - \{0\}\}, \quad (1.8)$$

where $Z(h_U)$ denotes the set of zeroes of the function $h_U(k)$. For any non-zero root of $h_U(k)$ the multiplicity $d_U(k^2)$ of the corresponding eigenvalue is

$$\forall k \in Z(h_U) - \{0\} : \quad d_U(k^2) = \text{Res} \left(\frac{d}{dz} \log(h_U(z)) \right) \Big|_{z=k}. \quad (1.9)$$

Let us mention, that the bound states of a given selfadjoint extension $\Delta_U \in \mathcal{M}$ are given by zeroes of $h_U(z)$ of the form $k = i\kappa$ with $\kappa > 0$, i.e. $k^2 < 0$. Furthermore, note that from eq. (1.7) it is easy to see that $\lim_{k \rightarrow 0} h_U(k) = 0$. This fact does not ensure that the corresponding selfadjoint extension Δ_U admits zero modes. The question about which selfadjoint extensions of \mathcal{M}_F admit zero modes will be solved in the next section.

Once all the selfadjoint extensions of \mathcal{M} have been explicitly characterized using the AIM formalism (see [3] for details), following [1] we can characterize all the selfadjoint extensions that belong to \mathcal{M}_F and hence that give rise to strongly consistent quantum field theories². One of the main results in [1] is the characterization of the set $\mathcal{M}_F \subset \mathcal{M}$ of non negative selfadjoint extensions $\forall L \in (0, \infty)$ (“strong consistency lemma”):

$$\mathcal{M}_F = \{U(\alpha, \beta, \mathbf{n}) \in \text{U}(2) = \mathcal{M} / \quad 0 \leq \alpha \pm \beta \leq \pi\}. \quad (1.10)$$

In Figure 1 we can see a representation of the set \mathcal{M}_F in the $\alpha\beta$ -plane. Whereas extensive results on the spectral zeta functions and the heat kernel are available for the standard boundary conditions like Dirichlet, Neumann, Robin or periodic [4, 5, 6, 7, 8], general boundary conditions as described in (1.4) have not been analyzed in comparable detail. This is the topic of the current paper. Generic interest in the analysis of spectral functions stems from their relevance in global analysis [5, 9] and quantum field theory topics such as the Casimir effect [10, 11, 12, 13, 14, 15, 16, 17].

The paper is organized as follows. In Section 2 we will answer the question which selfadjoint extensions within the strongly consistent extensions allow for zero modes. This is necessary as the details of the zeta function analysis depend on this input. Based upon the function $h_U(k)$, eq. (1.7), a contour integral representation of the zeta function for any strongly consistent selfadjoint extension will be derived. As usual, residues and certain values of the zeta function determine the associated heat kernel coefficients. The cases with and without zero modes are

²Given that the AIM formalism (first AIM theorem in [3, 1]) ensures the one-to-one correspondence between selfadjoint extensions of Δ over $[0, L]$ and unitary matrices of $\text{U}(2)$ from now on we will not make a distinction between self adjoint extensions $\Delta_U \in \mathcal{M}$ and unitary matrices $U \in \text{U}(2)$.

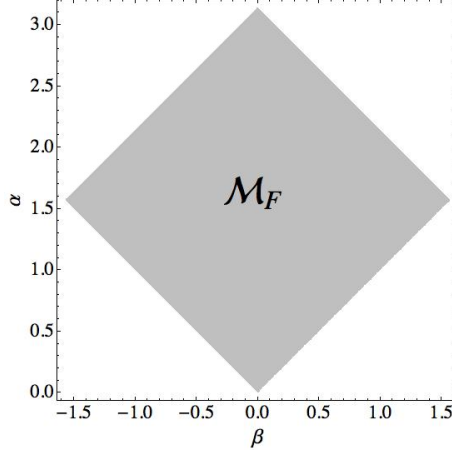


Figure 1: This graphic shows the set \mathcal{M}_F in the $\alpha\beta$ -plane. In the top corner of the rhombus are placed the Dirichlet boundary conditions, the bottom corner corresponds to Neumann boundary conditions meanwhile the left and right corners correspond to periodic (left for $n_1 = 1$ and right for $n_1 = -1$) and anti-periodic (left for $n_1 = -1$ and right for $n_1 = 1$).

treated in different subsections of Section 3. Results for standard boundary conditions are verified as a check. In Section 4 we use the integral representation of the zeta function to compute its derivative at $s = 0$, once again for all possible cases. Checks for known results are provided. In the conclusions we summarize the most important aspects of our work together with possible future directions of research.

2 Zero modes of $\Delta_U \in \mathcal{M}_F$

The purpose of this section is to study the zero mode structure of selfadjoint extensions contained in \mathcal{M}_F . In particular we will focus our attention on two main questions:

- Characterize the subset $\mathcal{M}_F^{(0)} \subset \mathcal{M}_F$ of selfadjoint extensions that have zero modes,

$$\mathcal{M}_F^{(0)} \equiv \{\Delta_U \in \mathcal{M}_F / 0 \in \sigma(\Delta_U)\}. \quad (2.1)$$

- Study the zero mode structure and compute $\dim(\ker \Delta_U)$ of any $\Delta_U \in \mathcal{M}_F^{(0)}$.

The motivation to study these two questions about the zero modes of the selfadjoint extensions contained in \mathcal{M}_F is to obtain a correct result of the $a_{1/2}$ heat kernel coefficient, for which we must know explicitly $\dim(\ker \Delta_U)$. There are no contributions of zero modes to residues of the zeta function.

The differential equation for the zero modes is

$$\frac{d^2}{dx^2} \psi_0(x) = 0, \quad (2.2)$$

and its general solution is given by

$$\psi_0(x) = a + bx, \quad (2.3)$$

where a and b are complex constant numbers. Notice that:

- When Δ is defined over the whole real line, the only solution to eq. (2.2) given by (2.3) with finite \mathcal{L}^2 norm is given by $a = b = 0$. Hence when Δ is defined over the real line there are no zero modes.
- On the other hand, when Δ is defined over the finite line $[0, L]$, due to the finite length of the interval the general solution (2.3) has always finite \mathcal{L}^2 norm. Hence when Δ is defined over the finite interval there exists the possibility of having constant and linear zero modes.

Given a selfadjoint extension $\Delta_U \in \mathcal{M}_F$, in order to decide if it admits zero modes of the general form (2.3) we must impose over (2.3) the corresponding boundary condition given by (1.1). From eq. (1.3) we obtain for $\psi_0(x)$

$$\varphi_{\pm}(\psi_0) \equiv \begin{pmatrix} a \mp ib \\ a + b(L \pm i) \end{pmatrix} = \begin{pmatrix} 1 & \mp i \\ 1 & L \pm i \end{pmatrix} \cdot \begin{pmatrix} a \\ b \end{pmatrix}. \quad (2.4)$$

Using (2.4) in the boundary condition (1.4) we obtain the linear system

$$\left[\begin{pmatrix} 1 & i \\ 1 & L - i \end{pmatrix} - U \cdot \begin{pmatrix} 1 & -i \\ 1 & L + i \end{pmatrix} \right] \cdot \begin{pmatrix} a \\ b \end{pmatrix} = 0. \quad (2.5)$$

This linear system is nothing else than the boundary condition for the zero modes. Given its importance in this section, let us call the matrix of the linear system D_U :

$$D_U = \begin{pmatrix} 1 & i \\ 1 & L - i \end{pmatrix} - U \cdot \begin{pmatrix} 1 & -i \\ 1 & L + i \end{pmatrix}. \quad (2.6)$$

Next we investigate the solutions of the linear system (2.5).

2.1 The first question: characterization of $\mathcal{M}_F^{(0)}$

From basic algebra we know that $\Delta_U \in \mathcal{M}_F$ will admit zero modes if and only if the linear system (2.5) has non-trivial solutions, i.e.

$$\ker(\Delta_U) \neq 0 \Leftrightarrow \ker(D_U) \neq 0 \Leftrightarrow \det(D_U) = 0. \quad (2.7)$$

Hence the characterization of $\mathcal{M}_F^{(0)}$ is given by

$$\mathcal{M}_F^{(0)} = \{U \in \mathcal{M}_F / \det(D_U) = 0\}. \quad (2.8)$$

To explicitly compute all the selfadjoint extensions contained in $\mathcal{M}_F^{(0)}$ we need to solve the secular equation of the linear system (2.5)

$$\det(D_U) = 0. \quad (2.9)$$

Introducing the parametrization (1.5) in (2.6) and simplifying we obtain

$$\det(D_U) = 2e^{i\alpha} [L (\cos(\alpha) - \cos(\beta)) - 2 (\sin(\alpha) + n_1 \sin(\beta))]. \quad (2.10)$$

Therefore neglecting the global factor $2e^{i\alpha}$ that is never zero the equation to solve is

$$L(\cos(\alpha) - \cos(\beta)) - 2(\sin(\alpha) + n_1 \sin(\beta)) = 0 \quad (2.11)$$

with the restrictions ensuring that the corresponding solution gives a matrix U that is in \mathcal{M}_F :

$$n_1 \in [-1, 1]; \alpha \in [0, \pi]; \beta \in [-\pi/2, \pi/2]; 0 \leq \alpha \pm \beta \leq \pi. \quad (2.12)$$

The simplest way to solve (2.11) is by imposing

$$\cos \alpha - \cos \beta = 0 \implies \alpha = \pm \beta,$$

which makes

$$\sin \alpha + n_1 \sin \beta = \sin \alpha \pm n_1 \sin \alpha = 0 \implies n_1 = \mp 1$$

necessary. In fact, these are all possible solutions. Because if $\cos \alpha - \cos \beta \neq 0$, we have

$$L = \frac{2 \sin \alpha + n_1 \sin \beta}{\cos \alpha - \cos \beta}$$

with $L > 0$. However, with the parameters confined by the conditions in (2.12) one can show that the right hand side is always negative. As a consequence we have shown that all the solutions to (2.11) that satisfy conditions (2.12) are given by

$$\mathcal{M}_F^{(0)} = \{U \in \mathcal{M}_F / n_1 = \pm 1; \alpha \in [0, \pi/2]; \beta = -n_1 \alpha\}. \quad (2.13)$$

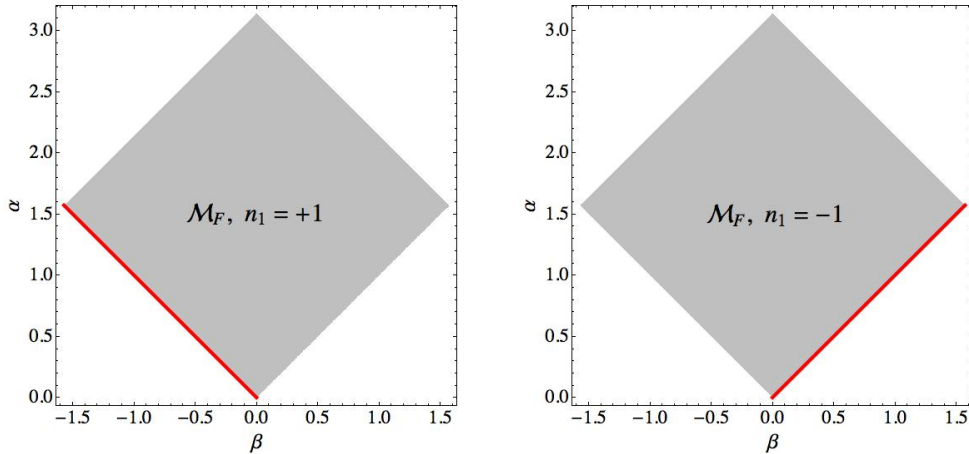


Figure 2: Representation of $\mathcal{M}_F^{(0)}$ (red lines) over the $\alpha\beta$ -plane

In terms of the parametrization given in (1.5) the unitary matrices contained in $\mathcal{M}_F^{(0)}$ are given by

$$U \in \mathcal{M}_F^{(0)} \implies U = e^{i\alpha} [\cos(\alpha)\mathbb{I} - in_1 \sin(\alpha)\sigma_1]; \alpha \in [0, \pi/2]; n_1 \in \{-1, 1\}. \quad (2.14)$$

Using the expression above for the matrices contained in $\mathcal{M}_F^{(0)}$ and the definition (2.6), we find

$$D_U = \begin{pmatrix} 0 & ie^{i\alpha} (2 \cos(\alpha) + L \sin(\alpha)) \\ 0 & -ie^{i\alpha} (2 \cos(\alpha) + L \sin(\alpha)) \end{pmatrix} \quad \forall U \in \mathcal{M}_F^{(0)}. \quad (2.15)$$

As can be seen from this expression above, when $U \in \mathcal{M}_F^{(0)}$ the matrix D_U has indeed zero determinant.

2.2 The second question: $\dim(\ker(\Delta_U))$ for $\Delta_U \in \mathcal{M}_F^{(0)}$

Taking into account eq. (2.5) and the meaning of the constants a and b , see eq. (2.3), the second question will be answered by studying the explicit solutions to (2.5) when D_U is given by expression (2.15), i.e. $U \in \mathcal{M}_F^{(0)}$. We will answer this second question in two lemmas with their corresponding demonstrations.

Lemma 1. *Any selfadjoint extension $\Delta_U \in \mathcal{M}_F^{(0)}$ admits a constant zero mode.*

Proof. To proof the lemma we only need to demonstrate that the column vector

$$v_c^{(0)} = \begin{pmatrix} a \\ 0 \end{pmatrix}, \quad a \neq 0, \quad (2.16)$$

belongs to $\ker(D_U)$ for any $U \in \mathcal{M}_F^{(0)}$ (notice that according to (2.3) when $b = 0$ and $a \neq 0$ the expression gives rise to the constant function over the interval $[0, L]$). For any $U \in \mathcal{M}_F^{(0)}$ the associated matrix D_U is given by (2.15). Since the first column in (2.15) is identically zero by direct trivial calculation

$$D_U \cdot v_c^{(0)} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \forall U \in \mathcal{M}_F^{(0)}. \quad (2.17)$$

Therefore $v_c^{(0)}$ is a solution to the linear system (2.5). Hence taking (2.3) into account for any $\Delta_U \in \mathcal{M}_F^{(0)}$ there exists a constant zero mode. ■

This lemma ensures that any selfadjoint extension $\Delta_U \in \mathcal{M}_F^{(0)}$ has at least a constant zero mode, i.e.

$$\forall \Delta_U \in \mathcal{M}_F^{(0)}, \quad \dim(\ker(\Delta_U)) = \dim(\ker(D_U)) \geq 1. \quad (2.18)$$

Since any $\Delta_U \in \mathcal{M}_F^{(0)}$ has a constant zero mode the only possibility to be explored now is the possibility of having selfadjoint extensions $\Delta_U \in \mathcal{M}_F^{(0)}$ that also admit a linear zero mode. The condition for a selfadjoint extension $\Delta_U \in \mathcal{M}_F^{(0)}$ to admit a linear zero mode is given by

$$\dim(\ker(\Delta_U)) = \dim(\ker(D_U)) = 2. \quad (2.19)$$

Since D_U is a 2×2 complex matrix

$$\dim(\ker(D_U)) = 2 \iff D_U = 0. \quad (2.20)$$

This condition ensures the existence of a linear zero mode for any selfadjoint extension $\Delta_U \in \mathcal{M}_F^{(0)}$ by the following argumentation:

- i. For any $\Delta_U \in \mathcal{M}_F^{(0)}$ there is a constant zero mode $\Rightarrow v_c^{(0)}$ given by (2.16) belongs to $\ker(D_U)$ for any $\Delta_U \in \mathcal{M}_F^{(0)}$.
- ii. $\Delta_U \in \mathcal{M}_F^{(0)}$ will admit a linear zero mode if and only if the matrix D_U is such that there exists in addition to $v_c^{(0)}$ a solution to the linear system (2.5) with $b \neq 0$ (see eq. (2.3)).

iii. Hence $\Delta_U \in \mathcal{M}_F^{(0)}$ will admit a linear zero mode if and only if

$$\dim(\ker(D_U)) = 2 \iff D_U = 0, \quad (2.21)$$

because any solution to (2.5) with $b \neq 0$ will be linearly independent of the vector $v_c^{(0)} \in \ker(D_U) \forall \Delta_U \in \mathcal{M}_F^{(0)}$.

Lemma 2. *There are no selfadjoint extensions $\Delta_U \in \mathcal{M}_F^{(0)}$ that admit a linear zero mode.*

Proof. Given any $\Delta_U \in \mathcal{M}_F^{(0)}$ the necessary and sufficient condition to admit a linear zero mode is (2.20). Since for $\Delta_U \in \mathcal{M}_F^{(0)}$ the associated D_U matrix is given by (2.15) the condition $D_U = 0$ is given by the equation

$$2 \cos(\alpha) + L \sin(\alpha) = 0 \Rightarrow \tan(\alpha) = -2/L. \quad (2.22)$$

Because L is the length of the interval $-2/L \leq 0$. Therefore there is no $\alpha \in [0, \pi/2]$ satisfying (2.22)³. Therefore no $\Delta_U \in \mathcal{M}_F^{(0)}$ can satisfy the condition $D_U = 0$, i.e. no $\Delta_U \in \mathcal{M}_F^{(0)}$ admits a linear zero mode. ■

To conclude this section we compile all the results in the following theorem.

Theorem 1. *The space $\mathcal{M}_F^{(0)} \subset \mathcal{M}_F$ of non-negative selfadjoint extensions of the Laplace operator Δ over $[0, L]$ that admit zero modes is given by*

$$\mathcal{M}_F^{(0)} = \{U \in \mathcal{M}_F / \quad n_1 = \pm 1; \quad \alpha \in [0, \pi/2]; \quad \beta = -n_1 \alpha\}. \quad (2.23)$$

In addition $\dim(\ker(\Delta_U)) = 1$ for any selfadjoint extension $\Delta_U \in \mathcal{M}_F^{(0)}$ and the unique zero mode is the constant function over the interval $[0, L]$.

3 The heat kernel expansion of $\Delta_U \in \mathcal{M}_F$

Using standard methods described for example in reference [7] we will next compute all the coefficients of the asymptotic expansion of the heat kernel corresponding to any selfadjoint extension $\Delta_U \in \mathcal{M}_F$. Before going over the explicit calculation let us introduce the general results contained in [7] that will be necessary in our calculation.

Let $\hat{\mathcal{O}}$ be an elliptic non-negative selfadjoint second order differential operator (in one dimension) over a Hilbert space \mathcal{H} . Let $f_{\hat{\mathcal{O}}}(z)$ be a holomorphic function over the complex plane such that

$$\lim_{k \rightarrow 0} f_{\hat{\mathcal{O}}} \neq 0, \infty, \quad (3.1)$$

and such that the non-zero part of the spectrum of $\hat{\mathcal{O}}$ is given by⁴

$$\tilde{\sigma}(\hat{\mathcal{O}}) = Z(f_{\hat{\mathcal{O}}}), \quad (3.2)$$

³Since $\Delta_U \in \mathcal{M}_F^{(0)}$ the angle α is restricted to lie in the interval $[0, \pi/2]$.

⁴We will denote by $\sigma(\hat{\mathcal{O}})$ the spectrum of the operator $\hat{\mathcal{O}}$ and $\tilde{\sigma}(\hat{\mathcal{O}})$ the non zero part of $\sigma(\hat{\mathcal{O}})$. Given a function $f(z)$ over the complex plane we will denote by $Z(f)$ the set of its zeroes over the complex plane.

where the multiplicities of eigenvalues are reflected in the order of the zeroes. When $f_{\hat{\mathcal{O}}}$ satisfies the conditions stated above, the spectral zeta function of the operator $\hat{\mathcal{O}}$ can be written as:

$$\zeta_{\hat{\mathcal{O}}}(s) = \frac{\sin(\pi s)}{\pi} \int_0^\infty dk \cdot k^{-2s} \partial_k \log(f_{\hat{\mathcal{O}}}(ik)). \quad (3.3)$$

The integral in (3.3) in the current context will be convergent in the region $1/2 < \Re s < 1$. However, expression (3.3) admits an analytical continuation to the whole complex plane with, in general, poles at

$$s = \frac{1}{2} - n; \quad n = 0, 1, 2, 3 \dots \quad (3.4)$$

The heat kernel coefficients can be computed in terms of the residues at the poles and the values at non-positive integers of $\zeta_{\hat{\mathcal{O}}}(s)$ [18]:

$$a_{1/2-z}(\hat{\mathcal{O}}) = \Gamma(z) \text{Res}(\zeta_{\hat{\mathcal{O}}}, s = z), \quad (3.5)$$

$$a_{1/2+q}(\hat{\mathcal{O}}) = (-1)^q \frac{\zeta_{\hat{\mathcal{O}}}(-q)}{\Gamma(q+1)} + \delta_{q,0} N_Z(\hat{\mathcal{O}}). \quad (3.6)$$

In eq. (3.6), $N_Z(\hat{\mathcal{O}})$ denotes the number of zero modes of the operator $\hat{\mathcal{O}}$.

Hence, according to formulas (3.5) and (3.6), in order to know all the heat kernel coefficients we only need to know the residues at the poles and the values at the non-positive integers of the spectral zeta function $\zeta_{\hat{\mathcal{O}}}(s)$. To use formula (3.3) we will need to use the secular equation given in formula (1.7). But directly from formula (1.7) it is easy to see that

$$\lim_{k \rightarrow 0} h_U(k) = 0. \quad (3.7)$$

Therefore using formula (3.3) to compute the residues and the values at the non-positive integers of $\zeta_U(s)$ for any $\Delta_U \in \mathcal{M}_F$ is not possible using the function (1.7) because it does not satisfy the condition (3.1). Hence we need to extract from (1.7) the suitable function by studying the behaviour of $h_U(z)$ as $z \rightarrow 0$.

3.1 Behaviour of $h_U(z)$ as $z \rightarrow 0$

If we perform power series expansion in k around $k = 0$ of the secular equation given by (1.7) up to first order in k we obtain

$$h_U(k) = 2ike^{i\alpha} (L(\cos(\alpha) - \cos(\beta)) - 2(n_1 \sin(\beta) + \sin(\alpha))) + O(k^2). \quad (3.8)$$

Taking into account eq. (2.10) for any $\Delta_U \in \mathcal{M}_F$ we can write the power series expansion above as

$$h_U(k) = ik \det(D_U) + O(k^2). \quad (3.9)$$

Hence for any $\Delta_U \in \mathcal{M}_F - \mathcal{M}_F^{(0)}$ the function that satisfies the required conditions to be used in the representation of the spectral zeta function given by eq. (3.3) is

$$\Delta_U \in \mathcal{M}_F - \mathcal{M}_F^{(0)} \quad \Rightarrow \quad f_U(k) = \frac{h_U(k)}{2ike^{i\alpha}}. \quad (3.10)$$

When the selfadjoint extension has zero modes ($\Delta_U \in \mathcal{M}_F^{(0)}$) the first order in k of the power expansion (3.9) is zero. Therefore we must expand h_U up to order 3 (notice from eq. (1.7) the function h_U is odd in k) to study the behavior at the origin:

$$\det(D_U) = 0 \Rightarrow h_U(k) = \frac{k^3 L}{3} (L (2 \sin(\alpha) - n_1 \sin(\beta)) + 3 (\cos(\alpha) + \cos(\beta))) + O(k^5). \quad (3.11)$$

Hence, in order to obtain the function that satisfies the conditions under which (3.3) is valid, we must divide by an extra k^2 when $\Delta_U \in \mathcal{M}_F^{(0)}$:

$$\Delta_U \in \mathcal{M}_F^{(0)} \Rightarrow f_U^{(0)}(k) = \frac{h_U(k)}{2ik^3 e^{i\alpha}}. \quad (3.12)$$

3.2 Heat kernel coefficients for $\Delta_U \in \mathcal{M}_F - \mathcal{M}_F^{(0)}$

For this case the appropriate function is given by eq. (3.10). Using (1.7) we can rewrite (3.10) for $k = ix$ as

$$\begin{aligned} f_U(ix) &= x e^{xL} \frac{\cos(\alpha) + \cos(\beta)}{2} \left[1 + \frac{2}{x} \frac{\sin(\alpha)}{\cos(\alpha) + \cos(\beta)} + \frac{1}{x^2} \frac{\cos(\beta) - \cos(\alpha)}{\cos(\alpha) + \cos(\beta)} \right. \\ &\quad - e^{-2xL} \left(1 + \frac{x^{-2}(\cos(\beta) - \cos(\alpha))}{\cos(\alpha) + \cos(\beta)} - \frac{2x^{-1} \sin(\alpha)}{\cos(\alpha) + \cos(\beta)} \right) \\ &\quad \left. + x^{-1} e^{-xL} \frac{4n_1 \sin(\beta)}{\cos(\alpha) + \cos(\beta)} \right]. \end{aligned} \quad (3.13)$$

Notice that the exponentially suppressed terms in (3.13) do not contribute to the poles and the values of $\zeta_{\Delta_U}(s)$ at the non-positive integers. Therefore, we can neglect them in the following formulas and just denote them as *e.s.t.* Hence $\log(f_U(ix))$ will be given by

$$\log(f_U(ix)) = \log\left(\frac{\cos(\alpha) + \cos(\beta)}{2}\right) + \log(x) + xL + \log(1 + \tau_U(x)), \quad (3.14)$$

where $\tau_U(x)$ is given by

$$\tau_U(x) = \frac{2}{x} \frac{\sin(\alpha)}{\cos(\alpha) + \cos(\beta)} + \frac{1}{x^2} \frac{\cos(\beta) - \cos(\alpha)}{\cos(\alpha) + \cos(\beta)} + e.s.t. \quad (3.15)$$

Now if we take into account the series expansion

$$\log(1 + \tau) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{\tau^n}{n}, \quad (3.16)$$

we can write

$$\log(1 + \tau_U(x)) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \left(\frac{2}{x} \frac{\sin(\alpha)}{\cos(\alpha) + \cos(\beta)} + \frac{1}{x^2} \frac{\cos(\beta) - \cos(\alpha)}{\cos(\alpha) + \cos(\beta)} \right)^n + e.s.t. \quad (3.17)$$

Using Newton's binomial formula we can write

$$\tau_U(x)^n/n = \sum_{j=0}^n \frac{\Gamma(n)2^{n-j} \sin^{n-j}(\alpha)}{\Gamma(j+1)\Gamma(n-j+1)} \frac{(\cos(\beta) - \cos(\alpha))^j}{(\cos(\alpha) + \cos(\beta))^n} x^{-(n+j)}. \quad (3.18)$$

After reordering the double summation we obtain

$$\log(1 + \tau_U(x)) = \sum_{m=1}^{\infty} b_m x^{-m}, \quad (3.19)$$

$$b_m \equiv \sum_{j=0}^{[m/2]} (-1)^{m-j+1} \frac{2^{m-2j} \Gamma(m-j) \sin^{m-2j}(\alpha)}{\Gamma(j+1)\Gamma(m-2j+1)} \frac{(\cos(\beta) - \cos(\alpha))^j}{(\cos(\alpha) + \cos(\beta))^{m-j}}, \quad (3.20)$$

where $m = 1, 2, 3, \dots$. Note that the coefficients b_m are not well defined when $\cos(\alpha) + \cos(\beta) = 0$. This specific case must be studied separately, and from now on we will assume that $\cos(\alpha) + \cos(\beta) \neq 0$. Hence, finally we obtain the following asymptotic series for $\partial_x \log(f_U(ix))$,

$$\partial_x \log(f_U(ix)) = L + x^{-1} - \sum_{m=1}^{\infty} m b_m x^{-m-1} + e.s.t. \quad (3.21)$$

Taking into account the integral representation (3.3) we can write for any selfadjoint extension $\Delta_U \in \mathcal{M}_F - \mathcal{M}_F^{(0)}$,

$$\zeta_{\Delta_U}(s) = \frac{\sin(\pi s)}{\pi} \int_0^1 dk \cdot k^{-2s} \partial_k \log(f_U(ik)) + \frac{\sin(\pi s)}{\pi} \int_1^{\infty} dk \cdot k^{-2s} \partial_k \log(f_U(ik)). \quad (3.22)$$

With this splitting all the information about the poles and the values of $\zeta_{\Delta_U}(s)$ at the non-positive integers is contained in the integration from 1 to ∞ . Therefore, in order to perform the analytic continuation of $\zeta_{\Delta_U}(s)$ to the complex plane, we have to perform the analytic continuation of

$$\frac{\sin(\pi s)}{\pi} \int_1^{\infty} dk \cdot k^{-2s} \partial_k \log(f_U(ik)) \quad (3.23)$$

to the complex plane. In order to do so we must remember the following identities:

$$\int_1^{\infty} dz \cdot z^{-2s} = \frac{1/2}{s - 1/2}, \quad (3.24)$$

$$\int_1^{\infty} dz \cdot z^{-2s-1} = \frac{1/2}{s}, \quad (3.25)$$

$$\int_1^{\infty} dz \cdot z^{-2s-m-1} = \frac{1/2}{s + m/2}. \quad (3.26)$$

Hence, the relevant information about the analytic continuation of (3.23) is contained in

$$\frac{\sin(\pi s)}{\pi} \int_1^{\infty} dk \cdot k^{-2s} \partial_k \log(f_U(ik)) = \frac{\sin(\pi s)}{\pi} \left(\frac{L/2}{s - 1/2} + \frac{1/2}{s} - \sum_{m=1}^{\infty} b_m \frac{m/2}{s + m/2} \right), \quad (3.27)$$

where in the bracket of the right hand side an analytic function of s has been neglected as it is irrelevant for our purposes in this section. Using this analytic continuation, the poles of $\zeta_{\Delta_U}(s)$ can be easily computed:

$$\begin{aligned} \text{res}(\zeta_{\Delta_U}(s), s = 1/2) &= \text{res}(L \sin(\pi s)/(2\pi(s - 1/2), s = 1/2) \\ \Rightarrow \text{res}(\zeta_{\Delta_U}(s), s = 1/2) &= \frac{L}{2\pi}, \end{aligned} \quad (3.28)$$

$$\begin{aligned} \text{res}\left(\zeta_{\Delta_U}(s), s = -\frac{(2n+1)}{2}\right) &= \text{res}\left(-\frac{(2n+1)b_{2n+1}\sin(\pi s)}{2\pi(s + (2n+1)/2)}, s = -\frac{(2n+1)}{2}\right) \\ \Rightarrow \text{res}\left(\zeta_{\Delta_U}(s), s = -\frac{(2n+1)}{2}\right) &= (-1)^n b_{2n+1} \frac{(2n+1)}{2\pi}, \quad n = 0, 1, 2, 3... \end{aligned} \quad (3.29)$$

Furthermore, it gives the values of $\zeta_{\Delta_U}(s)$ at the non-positive integers:

$$\zeta_{\Delta_U}(0) = \frac{1}{2} \lim_{s \rightarrow 0} \frac{\sin(\pi s)}{\pi s} = 1/2, \quad (3.30)$$

$$\zeta_{\Delta_U}(-n) = -nb_{2n} \lim_{s \rightarrow -n} \frac{\sin(\pi s)}{\pi(s+n)} = (-1)^{n+1} nb_{2n}, \quad n = 1, 2, 3... \quad (3.31)$$

Given eqs. (3.28)-(3.31), for any $\Delta_U \in \mathcal{M}_F - \mathcal{M}_F^{(0)}$ such that $\cos(\alpha) + \cos(\beta) \neq 0$ it is easy to compute the heat kernel coefficients using the general formulas (3.5) and (3.6). Namely, we find

$$a_0 = \frac{L}{2\sqrt{\pi}}, \quad a_{n+1} = -\frac{4^n n! b_{2n+1}}{(2n)! \sqrt{\pi}}, \quad n = 0, 1, 2, 3, ... \quad (3.32)$$

$$a_{1/2} = 1/2, \quad a_{n+1/2} = -\frac{b_{2n}}{(n-1)!}, \quad n = 1, 2, 3, ... \quad (3.33)$$

3.2.1 The case of $\Delta_U \in \mathcal{M}_F - \mathcal{M}_F^{(0)}$ with $\cos(\alpha) + \cos(\beta) = 0$

For this case the appropriate function reads

$$\begin{aligned} f_U^{(B)}(ix) &= e^{xL} \left[\sin(\alpha) + \frac{1}{2x} (\cos(\beta) - \cos(\alpha)) + e^{-xL} 2n_1 \sin(\beta) \right. \\ &\quad \left. + e^{-2xL} \left(\sin(\alpha) - \frac{1}{2x} (\cos(\beta) - \cos(\alpha)) \right) \right]. \end{aligned} \quad (3.34)$$

Following the same procedure as in the general case we expand, for $\alpha \neq \pi$,

$$\log(f_U^{(B)}(ix)) = \log(\sin(\alpha)) + xL + \sum_{m=1}^{\infty} c_m x^{-m} + e.s.t., \quad (3.35)$$

$$c_m = -\frac{\cotg^m(\alpha)}{m}. \quad (3.36)$$

Again the analytical continuation of

$$\frac{\sin(\pi s)}{\pi} \int_1^{\infty} dk \cdot k^{-2s} \partial_k \log(f_U(ik)) \quad (3.37)$$

provides the residues at the half integers and the values at the non-positive integers of $\zeta_{\Delta_U}^{(B)}(s)$,

$$\text{res} \left(\zeta_{\Delta_U}^{(B)}(s), s = 1/2 \right) = \frac{L}{2\pi}, \quad (3.38)$$

$$\text{res} \left(\zeta_{\Delta_U}^{(B)}(s), s = -\frac{(2n+1)}{2} \right) = (-1)^n c_{2n+1} \frac{(2n+1)}{2\pi}, \quad n = 0, 1, 2, 3, \dots, \quad (3.39)$$

$$\zeta_{\Delta_U}^{(B)}(0) = 0, \quad (3.40)$$

$$\zeta_{\Delta_U}^{(B)}(-n) = (-1)^{n+1} n c_{2n}, \quad n = 1, 2, 3, \dots \quad (3.41)$$

Once we use formulas (3.5) and (3.6) we obtain the corresponding heat kernel coefficients,

$$a_0^{(B)} = \frac{L}{2\sqrt{\pi}}, \quad a_{n+1}^{(B)} = -\frac{4^n n! c_{2n+1}}{(2n)! \sqrt{\pi}}, \quad n = 0, 1, 2, 3, \dots, \quad (3.42)$$

$$a_{1/2}^{(B)} = 0, \quad a_{n+1/2}^{(B)} = -\frac{c_{2n}}{(n-1)!}, \quad n = 1, 2, 3, \dots \quad (3.43)$$

Finally, the case $\alpha = \pi$, $\beta = 0$, has to be treated separately and

$$\partial_x \left(\ln f_U^{(B)}(ix) \right) \Big|_{\alpha=\pi} = L - \frac{1}{x} + e.s.t.$$

From here,

$$\text{res} \left(\zeta_{\Delta_U}^{(B)}(s) \Big|_{\alpha=\pi}, s = \frac{1}{2} \right) = \frac{L}{2\pi}, \quad \zeta_{\Delta_U}^{(B)}(0) \Big|_{\alpha=\pi} = -\frac{1}{2},$$

and

$$a_0^{(B)} \Big|_{\alpha=\pi} = \frac{L}{2\sqrt{\pi}}, \quad a_{1/2}^{(B)} \Big|_{\alpha=\pi} = -\frac{1}{2}, \quad (3.44)$$

with all other residues and relevant values respectively heat kernel coefficients equal to zero.

3.3 Heat kernel coefficients for $\Delta_U \in \mathcal{M}_F^{(0)}$

Taking into account (2.23) we can write (3.12) as

$$f_U^{(0)}(ix) = \frac{e^{xL}}{x} \cos(\alpha) \left[1 + \frac{\tan(\alpha)}{x} - 2 \frac{e^{-xL} \tan(\alpha)}{x} - e^{-2xL} \left(1 - \frac{\tan(\alpha)}{x} \right) \right]. \quad (3.45)$$

Therefore

$$\log \left(f_U^{(0)}(ix) \right) = \log(\cos(\alpha)) + xL - \log(x) + \sum_{n=1}^{\infty} \frac{(-1)^{n+1} \tan^n(\alpha)}{n} x^{-n} + e.s.t. \quad (3.46)$$

$$\Rightarrow \partial_x \log \left(f_U^{(0)}(ix) \right) = L - \frac{1}{x} + \sum_{n=1}^{\infty} (-1)^n \tan^n(\alpha) x^{-n-1} + e.s.t. \quad (3.47)$$

Hence the analytical continuation gives as before the required residues and values of $\zeta_{\Delta_U}^{(0)}(s)$,

$$\text{res} \left(\zeta_{\Delta_U}^{(0)}(s); s = 1/2 \right) = L/2\pi, \quad (3.48)$$

$$\text{res} \left(\zeta_{\Delta_U}^{(0)}(s); s = -\frac{2n+1}{2} \right) = \frac{(-1)^n}{2\pi} \tan^{2n+1}(\alpha), \quad n = 0, 1, 2, 3, \dots, \quad (3.49)$$

$$\zeta_{\Delta_U}^{(0)}(0) = -1/2, \quad (3.50)$$

$$\zeta_{\Delta_U}^{(0)}(-n) = \frac{1}{2}(-1)^n \tan^{2n}(\alpha), \quad n = 1, 2, 3, \dots \quad (3.51)$$

To obtain the heat kernel coefficients, we must take into account that there is one zero mode in all cases as demonstrated previously. Therefore we must add 1 to $a_{1/2}$:

$$a_0^{(0)} = \frac{L}{2\sqrt{\pi}}, \quad a_{1/2}^{(0)} = \frac{1}{2}, \quad (3.52)$$

$$a_{n+1}^{(0)} = -\frac{4^n n! \tan^{2n+1}(\alpha)}{(2n+1)! \sqrt{\pi}}, \quad n = 0, 1, 2, 3, \dots, \quad (3.53)$$

$$a_{n+1/2}^{(0)} = \frac{1}{2} \frac{\tan^{2n}(\alpha)}{n!}, \quad n = 1, 2, 3, \dots \quad (3.54)$$

The heat kernel coefficients obtained above for $\Delta_U \in \mathcal{M}_F^{(0)}$ become singular for $\alpha = \pi/2$. In this case instead

$$\partial_x \log \left(f_U^{(0)}(ix) \right) \Big|_{\alpha=\pi/2} = L - \frac{2}{x} + e.s.t., \quad (3.55)$$

and therefore the spectral zeta function $\zeta_{\Delta_U}^{(0)}(s) \Big|_{\alpha=\pi/2}$ will only have a residue at $s = 1/2$ and non zero value at $s = 0$,

$$\text{res} \left(\zeta_{\Delta_U}^{(0)}(s) \Big|_{\alpha=\pi/2}; s = 1/2 \right) = L/2\pi, \quad (3.56)$$

$$\zeta_{\Delta_U}^{(0)}(0) \Big|_{\alpha=\pi/2} = -1. \quad (3.57)$$

Hence the only non-vanishing heat kernel coefficients are given by

$$a_0^{(0)} \Big|_{\alpha=\pi/2} = \frac{L}{2\sqrt{\pi}}. \quad (3.58)$$

3.4 Heat kernel coefficients for common boundary conditions.

As a check of our calculations let us compare the results found for the heat kernel coefficients with the known ones for the most common boundary conditions.

- **Periodic boundary conditions.** The periodic boundary conditions are usually written as⁵

$$\psi(0) = \psi(L); \quad \psi'(0) = \psi'(L). \quad (3.59)$$

⁵When it is required that the solutions of the Laplace equation are smooth functions the periodic boundary conditions are given by the condition $\psi(0) = \psi(L)$. However square integrable solutions of the Laplace equation

Equivalently we can write the following two independent equations for periodic boundary conditions

$$\begin{aligned}\psi(0) + i\psi'(0) &= \psi(L) + i\psi'(L), \\ \psi(L) - i\psi'(L) &= \psi(0) - i\psi'(0).\end{aligned}$$

Hence following the notation of eq. (1.3) we can write the periodic boundary conditions in the form of (1.4) as

$$\varphi_-(\psi) = \sigma_1 \cdot \varphi_+(\psi), \quad (3.60)$$

being σ_1 the corresponding Pauli matrix. Therefore the unitary matrix that characterizes periodic boundary conditions is given by $U_p = \sigma_1 \in \mathcal{M}_F^{(0)} \Rightarrow \alpha = \pi/2, \beta = \pm\pi/2, n_1 = \mp 1$. The heat kernel coefficients are given by (3.58).

- **Dirichlet boundary condition.** The usual form of the Dirichlet boundary condition for any manifold M with boundary ∂M is

$$\psi|_{\partial M} = 0. \quad (3.61)$$

As can be seen the normal derivatives $\partial_n \psi|_{\partial M}$ do not enter in the boundary condition. From eq. (1.1) the general boundary condition for those unitary operators $U \in \mathcal{M}$ such that $1 \notin \sigma(U)$ can be written as

$$\psi|_{\partial M} = i \frac{\mathbb{I} + U}{\mathbb{I} - U} \cdot \partial_n \psi|_{\partial M}. \quad (3.62)$$

From this last expression, it is immediate to notice that the Dirichlet boundary condition is obtained when $U = -\mathbb{I}$. Therefore the Dirichlet boundary condition is given by $U_D = -\mathbb{I} \in \mathcal{M}_F - \mathcal{M}_F^{(0)} \Rightarrow \alpha = \pi, \beta = 0$. The heat kernel coefficients are given by (3.44).

- **Neumann boundary condition.** The usual form of the Neumann boundary condition for any manifold M with boundary ∂M is

$$\partial_n \psi|_{\partial M} = 0, \quad (3.63)$$

where ∂_n denotes the normal derivative to ∂M . As can be seen the boundary value $\psi|_{\partial M}$ does not enter in the boundary condition. From eq. (1.1) the general boundary condition for those unitary operators $U \in \mathcal{M}$ such that $-1 \notin \sigma(U)$ can be written as

$$\partial_n \psi|_{\partial M} = -i \frac{\mathbb{I} - U}{\mathbb{I} + U} \cdot \psi|_{\partial M}. \quad (3.64)$$

are not necessarily smooth. Therefore the condition $\psi(0) = \psi(L)$ does not necessarily give rise to periodic boundary conditions. As an example it is worth to mention the case of Dirac delta potentials (see references [19, 20, 21] for recent developments in the interpretation of Dirac delta potentials as boundary conditions and infinitely thin kinks) where the condition $\psi(0) = \psi(L)$ is satisfied but obviously the system does not satisfy periodic boundary conditions. Therefore in order to distinguish periodic boundary conditions from other types of point interactions it is necessary to include the second condition over the derivatives: $\psi'(0) = \psi'(L)$.

From this last expression it is immediate to notice that the Neumann boundary condition is obtained when $U = \mathbb{I}$. Therefore the Neumann boundary condition is given by $U_N = \mathbb{I} \in \mathcal{M}_F^{(0)} \Rightarrow \alpha = \beta = 0$. It is of note that in this case $\sin(\alpha) = 0$. Therefore from (3.52)-(3.54),

$$a_0^{(N)} = -\frac{L}{2\sqrt{\pi}}, \quad a_{1/2}^{(N)} = 1/2, \quad (3.65)$$

$$a_{n+1/2}^{(N)} = 0, \quad a_n^{(N)} = 0, \quad n = 1, 2, 3, \dots \quad (3.66)$$

- **Robin boundary conditions.** The common expression for the family of Robin boundary conditions is given by (see for example reference [22])

$$\psi|_{\partial M} - g \partial_n \psi|_{\partial M} = 0, \quad g \in (-\infty, \infty). \quad (3.67)$$

For the case in which the boundary manifold ∂M has several disjoint components $\partial M = \cup_i \Omega_i$ the family of Robin boundary conditions can be written as

$$\psi|_{\Omega_i} - g_i \partial_n \psi|_{\Omega_i} = 0, \quad g_i \in (-\infty, \infty). \quad (3.68)$$

The extreme values $g_i = 0, \infty$ correspond to Dirichlet and Neumann boundary conditions respectively in the i^{th} component of ∂M . Note that in the most general case the set of constants g_i do not have to be the same for all the disjoint components Ω_i of ∂M . For $M = [0, L]$ the boundary is formed by two points and therefore it has two disjoint components. The most simple choice of Robin boundary conditions in this case is

$$-\psi'(0) = \tan\left(\frac{\alpha}{2}\right) \psi(0), \quad \psi'(L) = \tan\left(\frac{\alpha}{2}\right) \psi(L), \quad \alpha \in [0, \pi]. \quad (3.69)$$

In a more compact notation we can write

$$\tan\left(\frac{\alpha}{2}\right) \psi|_{\partial M} - \partial_n \psi|_{\partial M} = 0, \quad \alpha \in [0, \pi]. \quad (3.70)$$

Taking into account eq. (3.64) and comparing it with expression (3.70) the unitary operator U_R for Robin boundary conditions satisfies the equation

$$\tan\left(\frac{\alpha}{2}\right) \mathbb{I} = -i \frac{\mathbb{I} - U_R}{\mathbb{I} + U_R}. \quad (3.71)$$

Therefore the unitary operator that characterizes the family of Robin boundary conditions given by (3.69) is given by $U_R = e^{i\alpha} \mathbb{I}$ as was firstly pointed out in references [23, 24]. Note that $U_R(\alpha = 0) = \mathbb{I} = U_N$ and $U_R(\alpha = \pi) = -\mathbb{I} = U_D$. In the parametrization (1.5) Robin boundary conditions correspond to $\beta = 0$. For $\alpha \in (0, \pi)$ $U_R(\alpha) \in \mathcal{M}_F - \mathcal{M}_F^{(0)}$ with $\cos(\alpha) + \cos(\beta) \neq 0$. Therefore the heat kernel coefficients for Robin boundary conditions are determined by eqs. (3.32) and (3.33). From eq. (3.20) it is easy to obtain the coefficients b_m for the Robin boundary conditions:

$$b_m^{(R)} = \tan^m\left(\frac{\alpha}{2}\right) \sum_{j=0}^{[m/2]} (-1)^{m-j+1} \frac{2^{m-2j} \Gamma(m-j)}{\Gamma(j+1) \Gamma(m-2j+1)}, \quad m = 1, 2, 3, \dots \quad (3.72)$$

Using now eqs. (3.32) and (3.33) it is immediate to compute the heat kernel coefficients for Robin boundary conditions to any desired order using any symbolic calculation software. As an example we show the first ten heat kernel coefficients:

$$a_0^{(R)} = L/2\sqrt{\pi}, \quad a_{1/2}^{(R)} = 1/2, \quad (3.73)$$

$$a_1^{(R)} = -\frac{2 \tan\left(\frac{\alpha}{2}\right)}{\sqrt{\pi}}, \quad a_2^{(R)} = -\frac{4 \tan^3\left(\frac{\alpha}{2}\right)}{3\sqrt{\pi}}, \quad (3.74)$$

$$a_3^{(R)} = -\frac{8 \tan^5\left(\frac{\alpha}{2}\right)}{15\sqrt{\pi}}, \quad a_4^{(R)} = -\frac{16 \tan^7\left(\frac{\alpha}{2}\right)}{105\sqrt{\pi}}, \quad (3.75)$$

$$a_{3/2}^{(R)} = \tan^2\left(\frac{\alpha}{2}\right), \quad a_{5/2}^{(R)} = \frac{1}{2} \tan^4\left(\frac{\alpha}{2}\right), \quad (3.76)$$

$$a_{7/2}^{(R)} = \frac{1}{6} \tan^6\left(\frac{\alpha}{2}\right), \quad a_{9/2}^{(R)} = \frac{1}{24} \tan^8\left(\frac{\alpha}{2}\right). \quad (3.77)$$

4 The functional determinant of Δ_U . Derivative at $s = 0$ of the spectral zeta function

In this section we compute the derivative of the zeta function at $s = 0$ for each of the different cases considered in Sections 3.2 and 3.3. As is well known, this derivative is a natural constituent when defining functional determinants of elliptic operators [9]. As usual we subtract and add back a suitable number of the asymptotic $k \rightarrow \infty$ terms of $\partial_k \log f_{\hat{\phi}}(ik)$ in (3.3). In the current context we have to subtract terms up to the order $1/k$ to make the integral well defined at $k = \infty$ once $s = 0$ is set. Introducing a small mass, that will be sent to zero at a suitable point of the computation, into our consideration, a presentation of (3.22) valid about $s = 0$ is given by

$$\begin{aligned} \zeta_{\Delta_U}(s) &= \frac{\sin \pi s}{\pi} \int_m^\infty dk (k^2 - m^2)^{-s} \partial_k \log \left[\frac{2f_U(ik)}{ke^{kL}(\cos \alpha + \cos \beta)} \right] \\ &\quad + \frac{\sin \pi s}{\pi} \int_m^\infty dk (k^2 - m^2)^{-s} \partial_k \log \left[ke^{kL} \frac{\cos \alpha + \cos \beta}{2} \right]. \end{aligned}$$

The integral in the first line by construction is analytic about $s = 0$ and its derivative at $s = 0$ is trivially computed. The needed integrals in the second line are known [25],

$$\begin{aligned} \int_m^\infty dk (k^2 - m^2)^{-s} &= \frac{m^{1-2s} \Gamma(1-s) \Gamma(s - \frac{1}{2})}{2\sqrt{\pi}}, \\ \int_m^\infty dk (k^2 - m^2)^{-s} \frac{1}{k} &= \frac{m^{-2s} \pi}{2 \sin \pi s}, \end{aligned}$$

and

$$\zeta'_{\Delta_U}(0) = -\log \left| \frac{2f_U(im)}{me^{mL}(\cos \alpha + \cos \beta)} \right| - Lm - \log m$$

is found. As $m \rightarrow 0$ we use

$$\lim_{m \rightarrow 0} f_U(im) = L(\cos \alpha - \cos \beta) - 2(\sin \alpha + n_1 \sin \beta)$$

to obtain

$$\zeta'_{\Delta_U}(0) = -\log \left| \frac{2L(\cos \alpha - \cos \beta) - 4(\sin \alpha + n_1 \sin \beta)}{\cos \alpha + \cos \beta} \right|. \quad (4.1)$$

The case treated in Section 3.2.1, for $\alpha \neq \pi$, follows along the same lines from

$$\begin{aligned} \zeta_{\Delta_U}^{(B)}(s) &= \frac{\sin \pi s}{\pi} \int_m^\infty dk (k^2 - m^2)^{-s} \partial_k \log \left[\frac{f_U^{(B)}(ik)}{e^{kL} \sin \alpha} \right] \\ &\quad + \frac{\sin \pi s}{\pi} \int_m^\infty dk (k^2 - m^2)^{-s} \partial_k \log [e^{kL} \sin \alpha]. \end{aligned}$$

In the limit as $m \rightarrow 0$ we obtain

$$\zeta_{\Delta_U}^{(B)'}(0) = -\log \left| \frac{L(\cos \alpha - \cos \beta) - 2(\sin \alpha + n_1 \sin \beta)}{\sin \alpha} \right|.$$

For $\alpha = \pi$, $\beta = 0$, instead we start with

$$\begin{aligned} \zeta_{\Delta_U}^{(B)}(s)|_{\alpha=\pi} &= \frac{\sin \pi s}{\pi} \int_m^\infty dk (k^2 - m^2)^{-s} \partial_k \log \left[\frac{f_U(ik)|_{\alpha=\pi}}{e^{kL}} \right] \\ &\quad + \frac{\sin \pi s}{\pi} \int_m^\infty dk (k^2 - m^2)^{-s} \partial_k \log \left[\frac{e^{kL}}{k} \right] \end{aligned}$$

to find

$$\zeta_{\Delta_U}^{(B)'}(0)|_{\alpha=\pi} = -\log(2L). \quad (4.2)$$

We are left to treat the cases with a zero mode dealt with in Section 3.3. There, for $\alpha \neq \pi/2$, the starting point is

$$\begin{aligned} \zeta_{\Delta_U}^{(0)}(s) &= \frac{\sin \pi s}{\pi} \int_m^\infty dk (k^2 - m^2)^{-s} \partial_k \log \left[\frac{f_U^{(0)}(ik)}{e^{kL} \cos \alpha} \right] \\ &\quad + \frac{\sin \pi s}{\pi} \int_m^\infty dk (k^2 - m^2)^{-s} \partial_k \log \left[\frac{e^{kL} \cos \alpha}{k} \right] \end{aligned}$$

leading to

$$\zeta_{\Delta_U}^{(0)'}(0) = -\log \left| \frac{L(2 \cos \alpha + L \sin \alpha)}{\cos \alpha} \right|.$$

For $\alpha = \pi/2$ instead

$$\begin{aligned}\zeta_{\Delta_U}^{(0)}(s)|_{\alpha=\pi/2} &= \frac{\sin \pi s}{\pi} \int_m^\infty dk (k^2 - m^2)^{-s} \partial_k \log \left[\frac{f_U^{(0)}(ik)|_{\alpha=\pi/2} k^2}{e^{kL}} \right] \\ &\quad + \frac{\sin \pi s}{\pi} \int_m^\infty dk (k^2 - m^2)^{-s} \partial_k \log \left[\frac{e^{kL}}{k^2} \right],\end{aligned}$$

leading to

$$\zeta_{\Delta_U}^{(0)'}(0) = -2 \log L. \quad (4.3)$$

These results can be confronted with the easily computed answers for periodic, Dirichlet and Neumann boundary conditions.

For Dirichlet boundary conditions the spectrum is $\lambda_n = (\pi n/L)^2$, $n \in \mathbb{N}$, with associated zeta function $\zeta_{Dir}(s) = (\pi/L)^{-2s} \zeta_R(2s)$. This gives $\zeta'_{Dir}(0) = -\log(2L)$ in agreement with (4.2).

For Neumann boundary conditions the spectrum is as above but with zero included. For the determinant the answers therefore again reads $\zeta'_{Neu}(0) = -\log(2L)$, which agrees with (4), once $\alpha = \beta = 0$ has been put.

Finally, for periodic boundary conditions the spectrum is $\lambda_n = (2\pi n/L)^2$, $n \in \mathbb{Z}$, with associated zeta function (zero mode excluded) $\zeta_{per}(s) = 2(2\pi/L)^{-2s} \zeta_R(2s)$. This shows $\zeta'_{per}(0) = -2 \log L$, again in agreement with (4.3).

As a specific new result, Robin boundary conditions as described above follow from (4.1) as

$$\zeta'_{\Delta_{U_R}}(0) = -\log \left(2 \tan \left(\frac{\alpha}{2} \right) \left(L \tan \left(\frac{\alpha}{2} \right) + 2 \right) \right).$$

5 Conclusions

In this article we have analyzed the spectral zeta function resulting from the Laplacian on the interval $[0, L]$ for the case when strongly consistent selfadjoint extensions are applied. Contour integral representations for the zeta functions are obtained for this class of selfadjoint extensions. These are used to compute leading heat kernel coefficients and the functional determinant in this context. Our results agree with known results for standard boundary conditions like Dirichlet, Neumann and periodic. The generalisation of these results to a scalar quantum field theory in $D + 1$ spacetime confined between two $D - 1$ dimensional plane parallel plates is straight forward for the heat kernel coefficients due to the factorization properties of the heat kernel in the same way as it is done in ref. [1].

The current article represents the start of further investigations into the details of heat kernel coefficients. Heat kernel coefficients are usually represented in terms of geometric invariants with universal multipliers depending on the boundary condition. The question arises how the multipliers depend on the chosen selfadjoint extension. In order to get some nontrivial boundary geometry involved a similar computation should be done for balls along the lines of [26, 27, 28], where choosing general selfadjoint extensions will lead to different combinations

of Bessel functions. Furthermore, following [29], surfaces of revolution are possible candidates to analyze how different selfadjoint extensions impact spectral functions.

Finally, the presented analysis could also be done for selfadjoint extensions that allow for finitely many negative eigenvalues by using a variation of the current procedure [30].

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